Analyzing Active Investment Strategies

Using tracking error variance decomposition.

Manuel Ammann, Stephan Kessler, and Jürg Tobler

For investment decisions it is important to know the investment strategies an asset manager uses. For mutual funds, for example, analysts such as Morningstar provide this information by classifying funds into categories reflecting specific classes of investment strategies. Such classifications are based mainly on information provided by the fund itself, on subjective judgment by the fund analyst, or on style analysis.[1]

We propose a decomposition of the non-central tracking error variance as an additional and an objective approach for identifying the investment strategies of all asset managers. It determines the cumulative extent of deviation from the asset manager's benchmark, indicating how actively the assets are managed.

Because this risk measure is computed from the squared return deviations between asset manager and benchmark, positive and negative returns are not averaged out as they are in return analysis. Non-central instead of the usual central tracking error variance allows us to recognize any consistent underperformance of the asset manager by analysis of tracking errors. Non-central tracking error variance reveals any increased risk-taking of a manager by leveraging.[2]

A specific investment strategy can be detected only if the model used for the tracking error variance decomposition captures its main characteristics. The decomposition can show to what extent the tracking error variance is a result of systematic deviations from the benchmark, reflecting a specific investment strategy, or whether the observed tracking error variance occurs primarily because of random deviations from the benchmark.

In practice, various models can be used to obtain reliable results. We apply two different decompositions. The first is based on the market model, which allows division

Manuel Ammann
is a professor of finance at the University of St. Gallen in St. Gallen, Switzerland.
manuel.ammann@unisg.ch

Stephan Kessler
is a research assistant at the University of St. Gallen.
stephan.kessler@unisg.ch

Jürg Tobler
is an investment manager at the Pension Fund of the City of Zurich.
juerg.tobler@pkzh.ch

of the average deviations from the benchmark into an alpha and a beta component. The second model extracts selection and timing activity relative to the benchmark. In a simulation study, we use the two models on returns, active returns, and tracking error variance to assess and compare the information content of these decompositions.[3] Other return models such as an asset pricing model with macroeconomic risk factors or the models by Fama and French [1993] or Carhart [1997] could also be used.

Our work contributes to the tracking error literature in three ways. First, it uses non-central tracking error variance as a key measure for the analysis of an active investment strategy. It applies well-known performance measurement models to the analysis of tracking error variance. While performance measurement focuses on the effects of an investment strategy on average excess returns, decomposition of the non-central tracking error variance is aimed solely at the detection of specific investment strategies.

Second, a simulation study shows how tracking error variance may be used to generate information that traditional methods are unable to uncover. Third, we assess the usefulness of the regression approach for the decomposition of tracking error variance to evaluate a manager's timing and selection abilities.

ANALYSIS OF TRACKING ERROR VARIANCE

The term *tracking error* refers to the imperfect replication of a given benchmark portfolio. Tracking errors can occur for a variety of reasons. The two most common are an attempt to outperform the benchmark by active portfolio management and the passive replication of the benchmark by a sampled portfolio. The main question in the case of active management is how much risk relative to the benchmark has to be taken to achieve the outperformance target. A tracking error measure gives an indication of this benchmark risk. For passive portfolio replication, tracking error measures are used to evaluate the success of the replicating portfolio strategy.

There are many different tracking error measures to control relative benchmark risk of portfolios or mutual funds. We focus our analysis on the non-central tracking error variance and propose two decompositions for it.

DEFINITION OF TRACKING ERROR VARIANCE

All tracking error measures are based on the return difference between a tracking portfolio and its benchmark. The difference between portfolio and benchmark returns over a certain investment period follows some distribution, which we call *tracking error distribution.* In finance, the standard deviation (i.e., the square root of the second central moment) of the tracking error distribution is referred to as the *tracking error.* Thus, tracking error variance might be understood in some contexts as the second central moment of this distribution. In the case of *tracking error variance*, we refer the second *non-central* moment of the tracking error distribution.

The tracking error variance is calculated according to

$$\hat{\tau}^2 = \frac{1}{n} \sum_{t=1}^{n} (r_{t,P} - r_{t,B})^2 \qquad (1)$$

where n is the number of observations and, $r_{t,P}$ and $r_{t,B}$ are past observed portfolio and benchmark returns, respectively.[4]

This is an estimate of the extent of the variation of the return of the tracking portfolio around the benchmark return during the estimation time period. Note that using non-central variance ensures that systematic over- or underperformance amplifies the risk measure. This is not the case for central variance, where the mean is deducted.

We use tracking error variance as an ex post measure for identifying investment strategies. Of course, tracking error variance can also be used ex ante to estimate portfolio risk. Discussions of the ex ante use of tracking error (variance) and some of the estimation problems in that context can be found in Jorion [2003], Kuenzi [2004], Lawton-Browne [2001], Satchell and Hwang [2001], and Scowcroft and Sefton [2001].

Regression Model

The decomposition of portfolio returns according to the regression model is obtained by regressing the returns on the benchmark return:[5]

$$r_P = \alpha + \beta r_B + \varepsilon \qquad (2)$$

where α is the expected uncorrelated outperformance (called the *alpha component*); βr_B is the return element correlated with the benchmark (called the *systematic component*); and ε is the unexpected uncorrelated outperformance. By the assumptions of the regression approach, the average of the latter component is 0.

As selection and timing are closely related to the alpha and beta parameters of the regression model, the results from that decomposition model can be analyzed with respect to their usefulness in identifying timing and selection. The active return (AR) decomposition is given as

$$r_P - r_B = \alpha + (\beta - 1)r_B + \varepsilon \tag{3}$$

In this decomposition $(\beta - 1)r_B$ is the outperformance that is related to the portfolio's correlation with the benchmark as measured by β. If $\beta > 1$, the outperformance is positive if the benchmark return is positive, and vice versa. This is the so-called *systematic* component. α and ε are interpreted as in the return case. For reporting purposes, the average of these two decompositions is computed.

The tracking error variance (TEV) can be decomposed into four terms (a detailed derivation is in the appendix):

$$\begin{aligned}\tau^2 &= \alpha^2 \\ &+(\beta-1)^2(\sigma_B^2+\mu_B^2) \\ &+\sigma_\varepsilon^2 \\ &+2\alpha(\beta-1)\mu_B\end{aligned} \tag{4}$$

where σ_B^2 is the variance of the benchmark return, σ_ε^2 is the variance of the regression residual, and μ_B is the expected benchmark return.

The first line in Equation (4) shows the portion of the TEV that arises from the generated alpha of the portfolio—the *alpha component* of the decomposition. The second line shows the part of the TEV that is caused by the deviation from the benchmark; we call this component the *systematic component*, since it is caused by the systematic risk factor. The third line shows the variance of a random component. It is attributable neither to the generated α nor to the systematic deviation from the benchmark portfolio. It is termed the *variance of the residual.* The last line shows a *cross-term* that is caused by the interaction of the generated α and the deviation from the benchmark.

This decomposition shows to what extent the outperformance is caused by correlation with the benchmark. For example, if the manager's strategy is to outperform the benchmark by leveraging without choosing a portfolio structure different from the benchmark composition, the regression will have the parameters: $\alpha = 0$, $\beta \neq 1$, and $\varepsilon = 0$. For passive index investing, the regression model returns $\alpha = 0$ and $\beta = 1$, identifying the applied strategy perfectly. Alternatively, if the portfolio structure diverges from the benchmark composition, the regression shows the presence and the extent of systematic out- or underperformance $(\alpha \neq 0)$ and how the exposure to the portfolio benchmark measured by β is changed.

An asset selection strategy under perfect foresight returns $\alpha > 0$, but nothing can be said about β. Perfect timing in switching between a riskless asset and a benchmark investment is difficult to detect as well, since it returns $\alpha > 0$ and $\beta > 0$, influencing several parts of the TEV decomposition. Therefore, the regression approach does not always identify timing and selection perfectly and unambiguously.

Simulation enables us to evaluate how the different abilities are captured in the regression decomposition. This is particularly interesting, since the regression approach is developed for a type of analysis other than to detect timing and selection. Simulation reveals whether the model is a useful alternative for timing and selection decomposition, which is more demanding in terms of input factors in identifying a manager's ability.

The regression approach can also be used to illustrate the ability of the non-central TEV to capture the effects of increased leverage. For this we rearrange the decomposition of TEV in Equation (4):

$$\begin{aligned}\tau^2 &= \alpha^2 + (\beta-1)^2\mu_B^2 + 2\alpha(\beta-1)\mu_B \\ &+(\beta-1)^2\sigma_B^2 \\ &+\sigma_\varepsilon^2\end{aligned} \tag{5}$$

In the first line, the portion of the TEV that arises from the expected outperformance, $\alpha + (\beta - 1)\mu_B$, is shown; we call this component *expected tracking error variance.* The second line shows the amount of TEV arising from benchmark deviation exposure, $\beta - 1$, and the third line is the TEV not explained by the regression model (residual TEV). The second and the third lines combined can be interpreted as random TEV.

The decomposition of tracking error variance in Equation (5) shows how the investment strategy, described by systematic outperformance $(\alpha \neq 0)$ and benchmark exposure $\beta \neq 1$, generates tracking error variance. The expected tracking error variance [first line of Equation (5)] is caused by the use of non-central instead of central TEV. It is the result of a higher-than-expected return of the fund, and it becomes apparent that deterministic systematic outperformance affects only expected tracking error variance, as is seen in (5). This increase in TEV is the same for riskless and risky investments. The higher TEV caused by the expected outperformance is not perceived as risk by an investor—deterministic outperformance is welcomed.

Exposure to a stochastic benchmark, on the other hand, not only generates expected outperformance, $(\beta - 1)\,\mu_B$, but also gives rise to random tracking error variance,

$(\beta-1)^2\sigma_B^2$. This effect is analogous to the well-known leverage effect; by leveraging a portfolio, a higher expected return (or return deviation) may be earned, but the variance of the portfolio (or return deviation) grows quadratically. This random TEV is a source of risk, as it is a stochastic deviation of the fund return from the benchmark return. This increased risk has to be compensated by a higher expected outperformance.

Some investors consider only the residual variance σ_ε^2. By ignoring the other two components of total tracking error variance, it is implicitly assumed that the expected outperformance $(\beta - 1)\mu_B$ is great enough to justify the additional tracking error variance $(\beta-1)^2\sigma_B^2$. An investor might not think this to be the case, or perhaps not for every β.

To put this differently, declaring residual tracking error variance as the relevant risk measure allows the manager to take on a lot of beta risk that is not controlled for and might not generate enough expected outperformance. Consequently, the use of σ_ε^2 alone for tracking error risk allows asset managers to manipulate the results of a performance evaluation because leverage is not controlled for. Therefore, non-central TEV has several advantages over the residual variance σ_ε^2.

Timing and Selection

A portfolio manager has essentially two ways of achieving outperformance, either by changing the structure of the portfolio, i.e., by overweighting assets expected to perform better than the benchmark and underweighting the other assets correspondingly (*selection*), or by leveraging the portfolio by changing the exposure to the benchmark return (β in the regression approach) without changing the portfolio structure (*benchmark timing*). A benchmark exposure greater than one implies that the portfolio will earn a higher return than the benchmark, given that the latter is positive.

Regression approaches such as proposed in Henriksson and Merton [1981] are designed to determine whether selection or timing activity generates statistically significant outperformance compared to the benchmark. As regressions focus on the averages of time series, however, such tests will signal no activity if a fund's selection activity happens to cancel out over the data sample. We therefore want a model that is able to detect any selection or timing activity. The model relies on the availability of data for the portfolio and benchmark composition for each time period and allows computation of a conditional tracking error variance.

The portfolio weights $n_{t,i}$ are determined as:

$$n_{t,i} = b_t m_{t,i} + d_{t,i} \tag{6}$$

where $m_{t,i}$ denotes the benchmark weights in period t. The parameter b_t is the same for all assets i and determines the fraction of the benchmark held in the portfolio. b_t can therefore be interpreted as a measure for benchmark timing. If $b_t > 1$, the portfolio is leveraged. The asset-specific parameters $d_{t,i}$ define the selection component of the portfolio in period t; in other words, $d_{t,i}$ determines the divergence of the portfolio weight of asset i from the corresponding benchmark weight apart from pure timing. By regressing in period t the weights $n_{t,i}$ on the benchmarkweights $m_{t,i}$ without intercept, we obtain the parameter b_t and $d_{t,i}$.

This model allows the computation of tracking error variance *for each time*, conditional on current portfolio weights, compared to our regression decomposition of TEV, which is an average over time of an unknown *time-constant* tracking error variance, as in Equation (1). Nonetheless, for reporting purposes it is useful to compute arithmetic averages of returns, active returns, and tracking error variances for the timing and selection decomposition as well. Using Equation (6) and

$$r_{t,P} = \sum_i n_{t,i}\, r_{t,i} \tag{7}$$

we obtain

$$E(r_{t,P}) = E(b_t r_{t,B}) + E(r_{t,S}) \tag{8}$$

where $r_{t,i}$ is the return of asset i, and $r_{t,S}$ denotes the return on the selection portfolio with weights $d_{t,i}$ in period t. In this way, the period t portfolio return is decomposed into a pure timing component $E\,(b_t r_{t,B})$ and a pure selection component $E\,(r_{t,S})$.

Next we construct active returns (AR) by subtracting the benchmark return from the portfolio return. Using Equations (6) and (7), we can represent the expected active return (AR) in period t by:

$$E(r_{t,P} - r_{t,B}) = E[(b_t - 1)r_{t,B}] + E\,(r_{t,S}) \tag{9}$$

Return and active return decomposition have two parts:

- The first terms ($E\,(b_t r_{t,B})$ and $E\,[(b_t - 1)r_{t,B}]$) in Equations (8) and (9) describe the part of the performance that is connected to the systematic market factor and is termed the *timing component.*

- The second term $[E(r_{t,S})]$ in Equations (8) and (9) describes the part of the return caused by a deviation from the market portfolio and is called the *selection component.*

The expectation of the conditional tracking error variance is given by

$$\begin{aligned}\tau^2 &= E[(b_t - 1)^2(\sigma_{t,B}^2 + \mu_{t,B}^2)] \\ &+E[\sigma_{t,S}^2 + \mu_{t,s}^2] \\ &+E[2(b_t - 1)(\sigma_{t,BS} + \mu_{t,B}\mu_{t,S})] \qquad (10)\end{aligned}$$

with

$$\sigma_{t,S}^2 = \sum_i \sum_j d_{t,i}\, d_{t,j}\, \sigma_{t,ij}$$

$$\mu_{t,S} = \sum_i d_{t,i}\, \mu_{t,i}$$

$$\sigma_{t,BS} = \sum_i d_{t,i}\, \sigma_{t,Bi}$$

where the expectation is estimated by calculating the arithmetic average across time. The formal derivation of this result appears in the appendix. This formula is used for the decomposition of (expected) tracking error variance in the simulation study. Again, the decomposition can be attributed to different activities of the manager:

- The first line in Equation (10) denotes the *timing component.*
- The second line mirrors the part of the return caused by a deviation from the benchmark—the *selection component.*
- The third line is the comoment between timing and selection, termed the *cross-term.*

This decomposition directly relates tracking error variance to the two basic abilities of a manager, timing and selection. The definition of timing and selection has one noteworthy implication. Since the two abilities are determined by a regression of the portfolio weights on the benchmark weights without intercept, $b_t > 0$ for all strategies. Thus, the decomposition is biased toward indicating more timing ability than the manager might have. Even a manager who uses 100% selection has at least some timing ability according to this decomposition.

One disadvantage of this approach is the data requirement. To carry out the analysis, we must know the weights of the assets in the fund. Today such data is available via databases like CDA/Spectrum Mutual Funds Holdings by Thomson Financial, and the analysis can be performed easily using standard computer software.

TRACKING ERROR VARIANCE ANALYSIS WITH SIMULATED ASSET RETURNS

We can investigate the performance of the tracking error variance decomposition using a simulation study to analyze simple selection and timing strategies. The parameters and the setup are chosen to approximate real-world asset markets. For simplicity we report only the results for a simulation with three assets. An analysis with up to 80 assets produces similar results. Stock prices are assumed to follow a geometric random walk. We construct an index serving as benchmark asset. Assets 1, 2, and 3 have weights of 20%, 30%, and 50%, respectively.

For the simulation, the assets are assumed to have a rate of return of $\mu = 0.05$, a volatility rate of $\sigma = 0.20$, and a correlation matrix of

$$C = \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{pmatrix}$$

For each asset, 20,000 returns are simulated. We analyze the performance of TEV in detecting simple investment strategies on the basis of these simulated asset returns. Results for five of these strategies are reported here. The investment strategies are designed to reflect basic selection and timing strategies.

The first and second strategy, denoted *best selection* and *best timing*, respectively, assume perfect foresight by the investor. The *best selection* strategy therefore always selects the asset that will perform best in the next time period. The *best timing* strategy invests in the benchmark asset whenever the benchmark earns a positive return and remains uninvested otherwise.[7]

The third strategy, denoted *random selection*, randomly invests all available capital in one of the three assets for one period at a time. At the end of the period, the dice are rolled again to determine the asset to be held in the next period. The fourth strategy, denoted *random timing*, either invests in the benchmark asset or remains uninvested, and the choice is made randomly and independently for each period.

Mixed strategies are designed to test the performance of the decomposition strategies under more realistic assumptions. Fund managers usually apply a mix of different investment strategies, such as a blend of timing and selection. The *mixed strategies* are designed by blending *perfect timing, perfect selection*, and *random selection*, thus modeling managers with different strengths and abilities.[8]

We start with analysis of the performance of the regression decomposition, which is simple to use in practice. Then we analyze the performance of the timing and selection model decompositions. Finally, we compare the performance of the two decompositions, and discuss whether the regression approach, which is less demanding in terms of inputs, is a useful substitute for the timing and selection decomposition in detecting a fund manager's abilities.

Regression Decomposition of Tracking Error Variance

The simulation study is applied to evaluate the performance of different strategy analyses in detecting the abilities of a fund manager. The advantage of the simulation approach is that the abilities of the fund manager are known, so we can test the performance of the different types of analyses (return, active return, and TEV decomposition) in characterizing the manager.

The results for the regression model are in Exhibit 1. The *best selection* case leads to high alpha, meaning most of the return and active return are attributable to this non-systematic element. This is consistent with attributing most of the TEV to alpha and the residual, i.e., to the parts uncorrelated with the benchmark. Overall, *best selection* results in by far the highest TEV. The AR decomposition performs best (12.15% of the 11.96% total active return), and the TEV decomposition returns the second-best results (0.0148 of total TEV of 0.0198).

Random selection results in a high variance of the residual, which makes up for 0.0142 of a total TEV of 0.0143. Thus, *random selection* is captured in the TEV decomposition mainly by the variance of the regression residual. The traditional return decomposition performs badly for the *random selection* strategy, showing a rather systematic investing strategy by falsely attributing 4.87 percentage points of the total performance of 4.90% to the systematic component. The active return decomposition gives no clear signals, attributing 0.03% and −0.19% to alpha and the systematic component, respectively.

Hence, the TEV decomposition is better in detecting selection activity. Random selection is identified by the high variance of the residual, while *best selection* is correctly identified by the alpha part of the decomposition (the latter is a benefit of using non-central TEV). A manager's skill in selection activity is detected by a high generated alpha.

Random timing, on the other hand, is more difficult to detect. Alpha and consequently the active return caused by alpha are close to zero (0.0007 and 0, respectively), which shows that there is no persistent non-systematic outperformance as in pure selection. Most of the negative expected tracking error is attributed to the systematic factor, further

Exhibit 1
Results of Regression Model in Simulation Analysis

			Best Selection	Best Timing	Random Selection	Random Timing	Mixed Strategy
Portfolio Return	Total	(2)	17.02%	9.50%	4.90%	2.64%	10.56%
	Alpha	(2)	12.15%	6.36%	0.03%	0.07%	6.90%
	Systematic	(2)	4.87%	3.14%	4.87%	2.57%	3.66%
Active Return	Total	(3)	11.96%	4.44%	-0.15%	-2.41%	5.50%
	Alpha	(3)	12.15%	6.36%	0.03%	0.07%	6.90%
	Systematic	(3)	-0.19%	-1.92%	-0.19%	-2.49%	-1.40%
Tracking Error	Total	(4)	0.0198	0.0083	0.0143	0.0149	0.0066
Variance	Alpha	(4.1)	0.0148	0.0040	0.0000	0.0000	0.0048
	Systematic	(4.2)	0.0000	0.0044	0.0000	0.0074	0.0023
	Var. of Resid.	(4.3)	0.0054	0.0023	0.0142	0.0076	0.0014
	Cross term	(4.4)	-0.0005	-0.0024	0.0000	0.0000	-0.0019

The mixed strategy models a manager investing 20% according to best selection, 70% according to best timing, and 10% according to random selection. The cross-terms are displayed for completeness, but cannot be interpreted reasonably and are thus not included in the discussion. The numbers in parentheses indicate the equation number and the line of that equation that is used to calculate the number in the particular row.

supporting the observation of timing activity. The randomness of timing leads to a TEV caused almost equally by the correlation of returns with the benchmark and the variance of the residual (0.00736 versus 0.00759, respectively). This finding is in contrast to random selection, where TEV is caused predominantly by the variance of the residual.

The *best timing* strategy is difficult to detect and leads to a much smaller total TEV than the *best selection* strategy. Just as for the *best selection* case, a large amount of the return (6.36 percentage points of the 9.50% total performance) and active return (6.36 percentage points of the 4.44% total performance) is attributed to the generated alpha. This is a misleading result since the systematic factor would be expected to exert more influence for this benchmark-oriented strategy.

The AR decomposition performs particularly badly, attributing a negative proportion to the systematic factor (−1.92 percentage points). Only the TEV decomposition shows a relatively large impact of 53.0% of the systematic factor on total TEV. As the timing component of the investment strategy increases, total TEV declines (compare, e.g., *best selection* and *best timing* with TEV of 0.0198 and 0.0083, respectively) and the TEV part attributed to the benchmark correlation becomes larger—in absolute terms as well as in relative terms. Thus, TEV is the superior indicator for good timing, resulting not only in the most accurate decomposition, but also indicating timing by a lower TEV.

So far we have considered only pure strategies. It is possible that the decompositions work differently when the strategies are mixed. Such mixed strategies model the world in a more realistic way, as managers always apply a mix of stock-picking and changing of the exposure to the broad market as well as random stock-picking (although no manager would admit this).

From the large number of different simulated strategies, we choose a strategy that relies heavily on timing for discussion, since it gives important insights. We model a manager who chooses 20% of assets by *best selection*, 70% by *best timing*, and 10% by *random selection*. Because return, active return, and tracking error variance are now determined by various factors, it might be more difficult for the measures to filter out the abilities of the manager.

This mixed strategy reveals that timing and selection are difficult to detect using return and active return decompositions. According to the return decomposition, the manager realizes about one-third of its returns with benchmark exposure (3.66 percentage points) and two-thirds with alpha generation (6.90 percentage points), making it impossible to identify the manager's strong timing ability. This occurs because the regression approach cannot detect pure timing.

The AR decomposition performs even more poorly by assigning a negative part of the AR to the systematic factor (−1.4% of 5.5% total AR). Only the TEV decomposition is able to detect the abilities of the manager reasonably well. *Random selection* is mirrored in the variance of the residual with a proportion of about 25% of TEV (0.0014 of 0.0066 total TEV). In the decompositions the part caused by the benchmark correlation is rather small, but with one-third of TEV (0.0023 of 0.0066 total TEV) correspondingly high.

The TEV decomposition can thus be used to obtain valuable and additional information on the abilities of the manager that cannot be gained by other means such as return or AR decomposition. When timing is present in the investment strategy, or random selection is used, or mixed strategies are applied, the TEV decomposition provides additional insights. The regression approach proves useful for strategy analysis, but needs to be applied carefully since timing effects are not easy to filter out. The value of the TEV decomposition is confirmed by its success in identifying random investing through a high variance of the residual.

Timing and Selection Decomposition

Applying simulation analysis to the timing and selection model decompositions, we show how tracking error variance can improve the evaluation of fund managers on the basis of a strategy analysis. The cross-term of the TEV decomposition is neglected in the discussion because it cannot be attributed accurately to either timing or selection.

Overall, a normal strategy analysis is able to detect the drivers of a strategy, as can be seen in Exhibit 2. The return decomposition in a timing and a selection component delivers good information on the true abilities of a manager, but it tends to overestimate the timing ability of a fund manager. For *best selection*, 30.4% of the positive return component (i.e., 5.17 percentage points of total expected return of 17.02%) is attributed to timing.

Decomposition of the active return (AR) delivers similar results as the return decomposition. But using active return, the proportions of timing or selection in active return are closer to the true ability of the manager. The AR ratio of timing to selection for managers that use *best selection* is about 1:100 (0.11% and 11.85% for timing and selection, respectively), showing that the manager is essentially a stock-picker. As in the return case, *best* and *random*

EXHIBIT 2
Results of Timing and Selection Model in Simulation Analysis

			Best Selection	Best Timing	Random Selection	Random Timing	Mixed Strategy
Portfolio Return	Total	(7)	17.02%	9.50%	4.90%	2.64%	10.56%
	Timing	(7)	5.17%	9.50%	4.46%	2.64%	8.13%
	Selection	(7)	11.85%	0.00%	0.44%	0.00%	2.44%
Active Return	Total	(8)	11.96%	4.44%	-0.15%	-2.41%	5.50%
	Timing	(8)	0.11%	4.44%	-0.60%	-2.41%	3.07%
	Selection	(8)	11.85%	0.00%	0.44%	0.00%	2.44%
Tracking Error Variance	Total	(10)	0.0249	0.0116	0.0191	0.0150	0.0065
	Timing	(10.1)	0.0037	0.0116	0.0037	0.0150	0.0066
	Selection	(10.2)	0.0232	0.0000	0.0160	0.0000	0.0011
	Cross-term	(10.3)	-0.0020	0.0000	-0.0006	0.0000	-0.0012

The mixed strategy models a manager investing 20% according to best selection, 70% according to best timing, and 10% according to random selection. The cross-terms are displayed for completeness, but cannot be interpreted reasonably and are thus not included in the discussion. The numbers in parentheses indicate the equation number and the line of that equation that is used to calculate the number in the particular row. The numbers are averages taken over the decompositions performed in every period studied.

timing are determined perfectly, allocating 100% of the active return to timing (4.44% and −2.41%, respectively). The TEV decomposition performs similarly to the AR decomposition, but the results are not as close to the true abilities of the manager. For example, for *best selection* 0.0232 of the TEV of 0.0269 (i.e., 86.2%) is allocated to selection, while for the AR decomposition it is 99.1%.[9]

Using the TEV decomposition, the *random timing* and the *best timing* strategy are determined perfectly. Thus, in the case of *best* and *random timing* strategies, the decompositions of all measures are able to identify the abilities of the managers perfectly and there is no superior measure.

The TEV is the best-performing measure if the manager follows a *random selection* strategy, i.e., has no selection ability. The return decomposition falsely signals a strong positive timing ability (4.46 percentage points of total return of 4.90%), while the AR decomposition is not so informative, indicating bad timing (−0.60%) and a somewhat weaker but positive selection ability (0.44%). Only the TEV is able to correctly attribute most of the fund performance to selection (0.0160 of 0.0191 total TEV).

As the purpose of the analysis is to distinguish managers with good investing skills from those without skills, this is an important feature. It occurs partially because the TEV is a second moment determined by squaring tracking errors. Consequently, positive and negative deviations cannot offset one another, as is the case for normal AR. Thus, in the *random selection* case, the AR is close to zero (−0.15%), since the benchmark has the same expected return as the *random selection* strategy, but the expected TEV is greater than zero (0.0191).

In a mixed strategy (20% *best selection*, 70% *best timing*, 10% *random selection*), the return decomposition performs well in characterizing the true abilities of the manager. It attributes 8.13% of total expected return to timing and 2.44% to selection. The active return does not perform as well, showing only a slightly stronger timing ability on active returns (3.07%) than selection (2.44%).

The TEV decomposition assigns 85.7% (0.0066) of the variation of tracking errors to timing and 14.3% (0.0011) to selection. Hence, it does not characterize the true abilities of the manager as well as the return decomposition, but still fares better than the method using AR. Results of other mixed strategies confirm these results, indicating that the results are robust.

Skillful and random investment strategies can be distinguished by combining the analysis of AR with the timing and selection decomposition of TEV (see Exhibit 2). In general, high expected ARs mean good abilities, and the TEV decomposition can be used to examine the managers' strengths more closely. A strong selection ability leads to high TEVs and a high proportion of selection. A strong timing ability, on the other hand, results in a much lower overall TEV but also a high timing proportion of TEV. In other words, AR is used to determine if there is ability (best versus random) and TEV to see the relative skills of the manager in investing (total timing versus total selection strategies).

The TEV decomposition in general allows a fairly good characterization of the manager, although in some rare circumstances the return or the AR decomposition perform slightly better. The TEV decomposition is the most stable measure, which never delivers completely wrong results, as the return and active return decomposition sometimes do. Thus, the TEV decomposition can improve strategy analysis and its reliability significantly. The TEV decomposition is particularly effective in the detection of *random selection.*

Regression Decomposition versus Timing and Selection Decomposition

We have shown failures in the identification of managers' true abilities using expected returns (failure for *random selection* using timing and selection decomposition and for *best timing* using the regression approach) and active return (failure for *random selection* using timing and selection as well as the regression approach and for *best timing* using the regression approach) with the timing and selection model as well as the regression model. Thus, decomposition of expected returns and ARs with either model regularly leads to misclassification.

Only for the decomposition of the TEV do we find no material defects in the simulation analysis. By delivering perfect identification (100% of a total TEV of 0.0116 and 0.0150 for *best* and *random timing*, respectively), the timing and selection approach is clearly dominant in the TEV decomposition of the *timing strategies.* The regression approach is less efficient in identifying timing (e.g., for *best timing*, a total TEV of 0.0083 is split into 0.0040 for the alpha, 0.0044 for the systematic variance, 0.0023 for the variance of the residual, and −0.0024 for the cross-term part).

For *best selection,* the regression model delivers a fairly good characterization of the asset manager's true abilities, although 0.0054 of a 0.0198 total TEV is attributed to the variance of the residual, indicating a fairly strong stochastic component not present in the asset manager's actions.[10] The timing and selection decomposition correctly allocates most of the total TEV of 0.0249 to selection (0.0232).

For *random selection,* the regression decomposition has an advantage by uncovering the randomness in the variance of the residual (0.0142 of 0.0143 total TEV), although it remains unclear if this is the result of *random timing* or *random selection.* Only the knowledge that *random timing* is captured by the systematic component (0.0074) and the variance of the residual (0.0142) alike allows the conclusion that the investor uses *random selection.* Rather, the timing and selection decomposition clearly indicates selection activity (0.0160 of 0.0191). Together with the low active return (−0.15%), this is a clear indication of random selection.

Finally, for the *mixed* strategy, the regression decomposition of TEV attributes the greatest part of the TEV (0.0048) to alpha, not reflecting the systematic component of this strategy (70% of the mixed strategy is *best timing*). The timing and selection decomposition of the *mixed strategy* is more precise in attributing 0.0066 of total TEV to timing and 0.0011 to selection. Thus, in total, a timing and selection decomposition of the tracking error variance delivers superior results.

The regression approach is interesting for analysts overall because it needs fewer inputs than the timing and selection decomposition. Another advantage is that *random selection* is mirrored accurately in the variance of the regression's residual (although timing is reflected in the alpha *and* the systematic part of the decompositions, complicating the identification of timing).

Detection of the different timing and selection strategies is easier and more precise using the timing and selection decomposition. The proportions of the timing and selection-induced parts of total TEV reflect the true abilities of managers more closely than measurement by the regression approach. In addition, the random strategies can be attributed more accurately to their respective category (i.e., *random timing* to timing and *random selection* to selection). These advantages justify the use of the more data-intensive timing and selection decomposition.

If no information about the asset weights is available, the simulation analysis shows that the regression approach is an informative alternative measure to the timing and selection decomposition of TEV for identifying timing and selection. The TEV decomposition delivers more robust and precise results than the decomposition of returns or active returns.

CONCLUSION

We have used simple performance measurement return models for the decomposition of non-central tracking error variance. For the exposition, we use two models: a regression decomposition and a timing selection decomposition. To investigate whether the tracking error variance decomposition provides useful information, we conduct a simulation study comparing tracking error variance decomposition with traditional return and active return decomposition.

The simulation study shows that tracking error variance decomposition is helpful in the performance evaluation of investment managers. For timing and selection decomposition, the tracking error variance analysis is superior and delivers important additional information. Most important, it lets us identify random behavior. Tracking error variance is particularly helpful when the regression approach is used and when the manager uses mixed strategies or random selection. The traditional decomposition of returns and active returns delivers misleading results when the investment manager has no skills or uses a mix of investment strategies.

The tracking error variance can also be decomposed using other performance measurement models. Multifactor models could be applied to characterize the abilities of an investment manager even more accurately.

APPENDIX

DERIVATIONS

We provide here proofs for the two different decomposition models.

Proof for the Regression Decomposition

Following the regression model for portfolio and benchmark returns, the tracking error variance from Equation (1) is given as:

$$\begin{aligned}\tau^2 &= E[(r_P - r_B)^2] \\ &= E[((\beta-1)r_B + \varepsilon^*)^2] \\ &= (\beta-1)^2 E(r_B^2) + E(\varepsilon^*)^2 + 2(\beta-1)E(r_B\varepsilon^*)\end{aligned}$$

Using $\sigma_{xy} = E(xy) - E(x)E(y)$ on each term gives:

$$\tau^2 = (\beta-1)^2(\sigma_B^2 + \mu_B^2) + \sigma_\varepsilon^2 + \alpha^2 + 2\alpha(\beta-1)\mu_B$$

The tracking error variance in Equation (5) can be derived as follows. By expressing the first and second moments of the portfolio return as functions of the benchmark return and the residual return, namely:

$$\begin{aligned}\mu_P &= \alpha + \beta\mu_B \\ \sigma_P^2 &= \beta^2\sigma_B^2 + \sigma_\varepsilon^2 \\ \sigma_{PB} &= \beta\sigma_B^2\end{aligned}$$

the tracking error variance can be expressed as:

$$\begin{aligned}\tau^2 &= E[(r_P - r_B)^2] \\ &= E(r_P^2) + E(r_B^2) - 2E(r_P\, r_B) \\ &= \sigma_P^2 + \mu_P^2 + \sigma_B^2 + \mu_B^2 - 2(\sigma_{PB} + \mu_P\mu_B) \\ &= \beta^2\sigma_B^2 + \sigma_\varepsilon^2 + (\alpha + \beta\mu_B)^2 \\ &\quad + \sigma_B^2 + \mu_B^2 - 2(\beta\sigma_B^2 + (\alpha+\beta\mu_B)\mu_B) \\ &= (\beta-1)^2\sigma_B^2 + \sigma_\varepsilon^2 + \alpha^2 + 2\alpha(\beta-1)\mu_B + (\beta-1)^2\mu_B^2 \\ &= \alpha^2 + (\beta-1)^2(\sigma_B^2 + \mu_B^2) + \sigma_\varepsilon^2 + 2\alpha(\beta-1)\mu_B\end{aligned}$$

This completes the proof.

Proof for the Timing and Selection Decomposition

Using the definition of the portfolio weights and the regression model for the asset returns, the portfolio return for period t can be written as

$$r_{t,P} = \sum_i n_{t,i}\, r_{t,i}$$

where $r_{t,i}$ is the return of asset i in period t. By Equation (6) we obtain

$$r_{t,P} = b_t \sum_i m_{t,i}\, r_{t,i} + \sum_i d_{t,i}\, r_{t,i} = b_t\, r_{t,B} + r_{t,S}$$

and

$$r_{t,P} - r_{t,B} = (b_t - 1)\, r_{t,B} + r_{t,S}$$

The conditional TEV for period t is then given as:

$$\begin{aligned}\tau_t^2 &= E_u[(r_{t,P} - r_{t,B})^2] \quad \text{for } u < t \\ &= E_u[((b_t - 1)r_{t,B} + r_{t,S})^2]\end{aligned}$$

where E_u is the expectation conditioned on information available at time u. Evaluating the expectation yields the formula:

$$\begin{aligned}\tau_t^2 =& (b_t - 1)^2(\sigma_{t,B}^2 + \mu_{t,B}^2) \\ &+ \sigma_{t,S}^2 + \mu_{t,S}^2 + 2(b_t - 1)(\sigma_{t,BS} + \mu_{t,B}\mu_{t,S})\end{aligned}$$

with

$$\sigma_{t,S}^2 = \sum_i \sum_j d_{t,i} d_{t,j} \sigma_{t,ij}$$

$$\mu_{t,S} = \sum_i d_{t,i} \mu_{t,i}$$

$$\sigma_{t,BS} = \sum_i d_{t,i} \sigma_{t,Bi}$$

For reporting purposes, the average of τ_t^2 across time is calculated to estimate the expected unconditional τ^2, and we arrive at

$$\tau^2 = E[\tau_t^2] = E[(b_t - 1)^2(\sigma_{t,B}^2 + \mu_{t,B}^2)] + E[\sigma_{t,S}^2 + \mu_{t,S}^2] + E[2(b_t - 1)(\sigma_{t,BS} + \mu_{t,B}\mu_{t,S})]$$

This completes the proof.

ENDNOTES

An earlier version of this article was titled "Measurement and Decomposition of Tracking Error Variance." The authors thank Bernd Brommundt, Michael Genser, Philipp Halbherr, Karl Keiber, Ralf Seiz, Michael Verhofen, Roger Walder, Daniel Wydler, and Rico von Wyss; seminar participants at University of St. Gallen; participants at the annual conference of the German Finance Association; and an anonymous referee for helpful comments.

[1]See, e.g., Sharpe [1992] and Wermers [2000].

[2]For a review of the literature on the identification of performance factors, see Ippolito [1993]. We do not address performance issues in connection with differences in investment strategies. While the decomposition of tracking error variance has been studied before (see, e.g., Vardharaj, Fabozzi, and Jones [2004]), the application of performance measurement models for the decomposition is new.

[3]Simulation studies in performance analysis have been used before. See, e.g., Ammann and Zimmermann [2001] or Kothari and Warner [2001].

[4]Because there is only one observable realization of $r_{t,P} - r_{t,B}$ for each time period t, we need to use time series data for the estimation. We assume that all $r_{t,P} - r_{t,B}$ within the estimation interval are identically and independently distributed (iid) so that the realizations of all $r_{t,P} - r_{t,B}$ in the estimation interval can be used to estimate the true τ_t^2. The effect of serial correlation in return deviations on tracking error estimation is examined in Pope and Yadav [1994].

[5]If the portfolio benchmark is the market portfolio, the regression model corresponds to Sharpe's [1963] market model.

[6]Alternative specifications show that the asset return specification is not critical to our analysis.

[7]Because we limit the investment universe to three assets, being uninvested means earning no return at all on the capital. Alternatively, a strategy could be devised that invests in the benchmark if the benchmark return exceeds the riskless rate and invests in the riskless asset otherwise.

[8]The *random selection* strategy used for the mixed strategies is different from the one used on a stand-alone basis. The stand-alone *random selection* approach chooses *one asset* randomly for 100% investment to compare the results to best selection. In the mixed case, *random selection* means that the investment *weights* are chosen randomly.

[9]A higher number of assets in the simulation does not affect the effectiveness of the TEV decomposition. Rather, the decomposition can produce even better results. If a similar simulation is performed using 80 assets instead of three, the return decomposition allocates 90.9% to selection, the AR decomposition 104.5% to selection, and the TEV decomposition 5.7% to timing, 106.6% to selection, and −12.3% to the cross-term. This effect can be attributed to the declining weight of single assets in the benchmark portfolio as the asset universe is enlarged because deviations from the benchmark by picking a small number of stocks for investment (i.e., applying a selection strategy) tend to be larger and easier to detect.

[10]When we stress-test this result by increasing the number of assets (up to 80) in the simulation, we find that the regression approach improves in identifing the *best selection* strategy.

REFERENCES

Ammann, Manuel, and Heinz Zimmermann. "Tracking Error and Tactical Asset Allocation." *Financial Analysts Journal*, 57(2) (2001), pp. 32-43.

Carhart, Mark M. "On Persistence of Mutual Fund Performance." *Journal of Finance*, 52 (1997), pp. 57-82.

Fama, Eugene F., and Kenneth French. "Common Risk Factors in the Returns of Stocks and Bonds." *Journal of Financial Economics*, 33 (1993), pp. 3-56.

Henriksson, Roy D., and Robert C. Merton. "On Market Timing and Investment Performance II: Statistical Procedures for Evaluating Forecasting Skills." *Journal of Business*, 54 (1981), pp. 513-533.

Ippolito, Richard A. "On Studies of Mutual Fund Performance: 1962-1991." *Financial Analysts Journal*, 49(1) (1993), pp. 42-50.

Jorion, Philippe. "Portfolio Optimization with Tracking-Error Constraints." *Financial Analysts Journal*, 59(5) (2003), pp. 70-82.

Kothari, S. P., and Jerold B. Warner. "Evaluating Mutual Fund Performance." *Journal of Finance*, 56(5) (2001), pp. 1985-2010.

Kuenzi, David E. "Tracking Error and the Setting of Tactical Ranges." *The Journal of Investing*, 13(1) (2004), pp. 35-44.

Lawton-Browne, Carola. "An Alternative Calculation of Tracking Error." *Journal of Asset Management*, 2 (2001), pp. 223-234.

Pope, Peter F., and Pradeep K. Yadav. "Discovering Errors in Tracking Error." *The Journal of Portfolio Management*, 20(2) (1994), pp. 27-32.

Satchell, Stephen E., and Soosung Hwang. "Tracking Error: Ex Ante versus Ex Post Measures." *Journal of Asset Management*, 2 (2001), pp. 241-246.

Scowcroft, Alan, and James Sefton. "Do Tracking Errors Reliably Estimate Portfolio Risk?" *Journal of Asset Management*, 2 (2001), pp. 205-222.

Sharpe, William F. "Asset Allocation: Management Style and Performance Measurement." *The Journal of Portfolio Management*, 18(2) (1992), pp. 7-19.

——. "A Simplified Model for Portfolio Analysis." *Management Science*, 9 (1963), pp. 277-293.

Vardharaj, Raman, Frank J. Fabozzi, and Frank J. Jones. "Determinants of Tracking Error for Equity Portfolios." *Journal of Investing*, 13(2) (2004), pp. 37-47.

Wermers. Russ. "Mutual Fund Performance: An empirical Decomposition in to stock-Picking Talent, Style, Transactions Costs, and Expenses." *Journal of Finance*, 55(4) (2000), pp. 1655-1695.